Supplemental Materials for "Causal Effects of Treatments for Informative Missing Data Due to Progression/Death"

Detailed Calculations for the Covariance Matrix of $SACE_{k,\cdot}(1,0)$

We first describe the calculation for the covariance matrix for the log of $SACE_{k,1}(1,0)$. The log of $SACE_{k,1}(1,0)$ is

$$\begin{aligned}\log \mathrm{SACE}_{k,1}(1,0) &= \log P(Y_T(1) > k | A_T(0)) - \log \{1 - P(Y_T(1) > k | A_T(0))\} \\ &\quad - \log P(Y_T(0) > k | A_T(0)) + \log \{1 - P(Y_T(0) > k | A_T(0))\}.\end{aligned}$$

The derivatives of $\log \mathrm{SACE}_k(1,0)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\begin{aligned}\frac{\partial \log SACE_{k,1}(1,0)}{\partial \beta_0(\mathrm{tx})} &= \frac{\partial \log SACE_{k,1}(1,0)}{\partial h_{T,\mathrm{tx}}(k)} \frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta_0(\mathrm{tx})}, \\ \frac{\partial \log SACE_{k,1}(1,0)}{\partial \beta(\mathrm{tx})} &= \frac{\partial \log SACE_{k,1}(1,0)}{\partial h_{T,\mathrm{tx}}(k)} \frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta(\mathrm{tx})}.\end{aligned}$$

$\frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta_0(\mathrm{tx})}$ and $\frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta(\mathrm{tx})}$ are given in Lee and Daniels (2007). The derivatives of $\log \mathrm{SACE}_{k,1}(1,0)$

with respect to ψ_0, ψ_1, ψ_2, $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$ are given by

$$\begin{aligned}
\frac{\partial \log SACE_{k,1}(1,0)}{\partial \psi_0} &= \frac{\frac{\partial P(Y_T(1)>k|A_T(0))}{\partial \psi_0}}{P(Y_T(1)>k|A_T(0))\left\{1-P(Y_T(1)>k|A_T(0))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(0))}{\partial \psi_0}}{P(Y_T(0)>k|A_T(0))\left\{1-P(Y_T(0)>k|A_T(0))\right\}}, \qquad (1)\\
\frac{\partial \log SACE_{k,1}(1,0)}{\partial \psi_1} &= \frac{\frac{\partial P(Y_T(1)>k|A_T(0))}{\partial \psi_1}}{P(Y_T(1)>k|A_T(0))\left\{1-P(Y_T(1)>k|A_T(0))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(0))}{\partial \psi_1}}{P(Y_T(0)>k|A_T(0))\left\{1-P(Y_T(0)>k|A_T(0))\right\}}, \qquad (2)\\
\frac{\partial \log SACE_{k,1}(1,0)}{\partial \psi_2} &= \frac{\frac{\partial P(Y_T(1)>k|A_T(0))}{\partial \psi_2}}{P(Y_T(1)>k|A_T(0))\left\{1-P(Y_T(1)>k|A_T(0))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(0))}{\partial \psi_2}}{P(Y_T(0)>k|A_T(0))\left\{1-P(Y_T(0)>k|A_T(0))\right\}}, \qquad (3)\\
\frac{\partial \log SACE_{k,1}(1,0)}{\partial h_{T,0}(k)} &= - \frac{\frac{\partial P(Y_T(0)>k|A_T(0))}{\partial h_{T,0}(k)}}{P(Y_T(0)>k|A_T(0))\left\{1-P(Y_T(0)>k|A_T(0))\right\}}, \qquad (4)\\
\frac{\partial \log SACE_{k,1}(1,0)}{\partial h_{T,1}(k)} &= \frac{\frac{\partial P(Y_T(1)>k|A_T(0))}{\partial h_{T,1}(k)}}{P(Y_T(1)>k|A_T(0))\left\{1-P(Y_T(1)>k|A_T(0))\right\}}, \qquad (5)\\
\frac{\partial \log SACE_{k,1}(1,0)}{\partial h_{T,2}(k)} &= 0,
\end{aligned}$$

where the numerators in (1)-(5) are calculated using numerical derivatives.

We now describe the calculation for the covariance matrix for the log of $SACE_{k,2}(1,0)$. The log of $SACE_{k,2}(1,0)$ is

$$\begin{aligned}
\log \mathrm{SACE}_{k,2}(1,0) &= \log P(Y_T(1)>k|A_T(4)) - \log\left\{1-P(Y_T(1)>k|A_T(4))\right\} \\
&\quad - \log P(Y_T(0)>k|A_T(4)) + \log\left\{1-P(Y_T(0)>k|A_T(4))\right\}.
\end{aligned}$$

The derivatives of $\log \mathrm{SACE}_{k,2}(1,0)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\begin{aligned}
\frac{\partial \log SACE_{k,2}(1,0)}{\partial \beta_0(\mathrm{tx})} &= \frac{\partial \log SACE_{k,2}(1,0)}{\partial h_{T,\mathrm{tx}}(k)} \frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta_0(\mathrm{tx})}, \\
\frac{\partial \log SACE_{k,2}(1,0)}{\partial \beta(\mathrm{tx})} &= \frac{\partial \log SACE_{k,2}(1,0)}{\partial h_{T,\mathrm{tx}}(k)} \frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta(\mathrm{tx})}.
\end{aligned}$$

The derivatives of $\log \mathrm{SACE}_{k,2}(1,0)$ with respect to ψ_0, ψ_1, ψ_2, $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$

are given by

$$\begin{aligned}
\frac{\partial \log SACE_{k,2}(1,0)}{\partial \psi_0} &= \frac{\frac{\partial P(Y_T(1)>k|A_T(4))}{\partial \psi_0}}{P(Y_T(1)>k|A_T(4))\left\{1-P(Y_T(1)>k|A_T(4))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(4))}{\partial \psi_0}}{P(Y_T(0)>k|A_T(4))\left\{1-P(Y_T(0)>k|A_T(4))\right\}}, \\
\frac{\partial \log SACE_{k,2}(1,0)}{\partial \psi_1} &= \frac{\frac{\partial P(Y_T(1)>k|A_T(4))}{\partial \psi_1}}{P(Y_T(1)>k|A_T(4))\left\{1-P(Y_T(1)>k|A_T(4))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(4))}{\partial \psi_1}}{P(Y_T(0)>k|A_T(4))\left\{1-P(Y_T(0)>k|A_T(4))\right\}}, \\
\frac{\partial \log SACE_{k,2}(1,0)}{\partial \psi_2} &= \frac{\frac{\partial P(Y_T(1)>k|A_T(4))}{\partial \psi_2}}{P(Y_T(1)>k|A_T(4))\left\{1-P(Y_T(1)>k|A_T(4))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(4))}{\partial \psi_2}}{P(Y_T(0)>k|A_T(4))\left\{1-P(Y_T(0)>k|A_T(4))\right\}}, \\
\frac{\partial \log SACE_{k,2}(1,0)}{\partial h_{T,0}(k)} &= -\frac{\frac{\partial P(Y_T(0)>k|A_T(4))}{\partial h_{T,0}(k)}}{P(Y_T(0)>k|A_T(4))\left\{1-P(Y_T(0)>k|A_T(4))\right\}}, \\
\frac{\partial \log SACE_{k,2}(1,0)}{\partial h_{T,1}(k)} &= \frac{\frac{\partial P(Y_T(1)>k|A_T(4))}{\partial h_{T,1}(k)}}{P(Y_T(1)>k|A_T(4))\left\{1-P(Y_T(1)>k|A_T(4))\right\}}, \\
\frac{\partial \log SACE_{k,2}(1,0)}{\partial h_{T,2}(k)} &= 0,
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial P(Y_T(j)>k|A_T(4))}{\partial \psi_{\text{tx}}} &= \frac{\tau \frac{\partial P(Y_T(j)>k|A_T(0))}{\partial \psi_{\text{tx}}}}{\left\{1+(\tau-1)P(Y_T(j)>k|A_T(0))\right\}^2}, \\
\frac{\partial P(Y_T(0)>k|A_T(4))}{\partial h_{T,0}(k)} &= \frac{\tau \frac{\partial P(Y_T(0)>k|A_T(0))}{\partial h_{T,0}(k)}}{\left\{1+(\tau-1)P(Y_T(0)>k|A_T(0))\right\}^2}, \\
\frac{\partial P(Y_T(1)>k|A_T(4))}{\partial h_{T,1}(k)} &= \frac{\tau \frac{\partial P(Y_T(1)>k|A_T(0))}{\partial h_{T,1}(k)}}{\left\{1+(\tau-1)P(Y_T(1)>k|A_T(0))\right\}^2},
\end{aligned}$$

for $j = 0, 1$ and $\text{tx} = 0, 1, 2$.
We now describe the calculation for the covariance matrix for the log of $SACE_{k,3}(1,0)$.
The log of $SACE_{k,3}(1,0)$ is

$$\begin{aligned}
\log \text{SACE}_{k,3}(1,0) &= \log P(Y_T(1)>k|A_T(0)\cup A_T(4)) - \log\left\{1-P(Y_T(1)>k|A_T(0)\cup A_T(4))\right\} \\
&\quad - \log P(Y_T(0)>k|A_T(0)\cup A_T(4)) + \log\left\{1-P(Y_T(0)>k|A_T(0)\cup A_T(4))\right\}.
\end{aligned}$$

The derivatives of $\log \mathrm{SACE}_{k,3}(1,0)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\frac{\partial \log SACE_{k,3}(1,0)}{\partial \beta_0(\mathrm{tx})} = \frac{\partial \log SACE_{k,3}(1,0)}{\partial h_{T,\mathrm{tx}}(k)} \frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta_0(\mathrm{tx})},$$
$$\frac{\partial \log SACE_{k,3}(1,0)}{\partial \beta(\mathrm{tx})} = \frac{\partial \log SACE_{k,3}(1,0)}{\partial h_{T,\mathrm{tx}}(k)} \frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta(\mathrm{tx})}.$$

The derivatives of $\log \mathrm{SACE}_{k,3}(1,0)$ with respect to ψ_0, ψ_1, ψ_2, $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$ are given by

$$\begin{aligned}
\frac{\partial \log SACE_{k,3}(1,0)}{\partial \psi_0} &= \frac{\frac{\partial P(Y_T(1)>k|A_T(0)\cup A_T(4))}{\partial \psi_0}}{P(Y_T(1)>k|A_T(0)\cup A_T(4))\left\{1-P(Y_T(1)>k|A_T(0)\cup A_T(4))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(0)\cup A_T(4))}{\partial \psi_0}}{P(Y_T(0)>k|A_T(0)\cup A_T(4))\left\{1-P(Y_T(0)>k|A_T(0)\cup A_T(4))\right\}}, \\
\frac{\partial \log SACE_{k,3}(1,0)}{\partial \psi_1} &= \frac{\frac{\partial P(Y_T(1)>k|A_T(0)\cup A_T(4))}{\partial \psi_1}}{P(Y_T(1)>k|A_T(0)\cup A_T(4))\left\{1-P(Y_T(1)>k|A_T(0)\cup A_T(4))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(0)\cup A_T(4))}{\partial \psi_1}}{P(Y_T(0)>k|A_T(0)\cup A_T(4))\left\{1-P(Y_T(0)>k|A_T(0)\cup A_T(4))\right\}}, \\
\frac{\partial \log SACE_{k,3}(1,0)}{\partial \psi_2} &= \frac{\frac{\partial P(Y_T(1)>k|A_T(0)\cup A_T(4))}{\partial \psi_2}}{P(Y_T(1)>k|A_T(0)\cup A_T(4))\left\{1-P(Y_T(1)>k|A_T(0)\cup A_T(4))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(0)\cup A_T(4))}{\partial \psi_2}}{P(Y_T(0)>k|A_T(0)\cup A_T(4))\left\{1-P(Y_T(0)>k|A_T(0)\cup A_T(4))\right\}}, \\
\frac{\partial \log SACE_{k,3}(1,0)}{\partial h_{T,0}(k)} &= -\frac{\frac{\partial P(Y_T(0)>k|A_T(0)\cup A_T(4))}{\partial h_{T,0}(k)}}{P(Y_T(0)>k|A_T(0)\cup A_T(4))\left\{1-P(Y_T(0)>k|A_T(0)\cup A_T(4))\right\}}, \\
\frac{\partial \log SACE_{k,3}(1,0)}{\partial h_{T,1}(k)} &= \frac{\frac{\partial P(Y_T(1)>k|A_T(0)\cup A_T(4))}{\partial h_{T,1}(k)}}{P(Y_T(1)>k|A_T(0)\cup A_T(4))\left\{1-P(Y_T(1)>k|A_T(0)\cup A_T(4))\right\}}, \\
\frac{\partial \log SACE_{k,3}(1,0)}{\partial h_{T,2}(k)} &= 0,
\end{aligned}$$

where

$$
\begin{aligned}
\frac{\partial P(Y_T(0) > k|A_T(0) \cup A_T(4))}{\partial \psi_0} &= \frac{1}{P(A_T(0) \cup A_T(4))} \left\{ \frac{\partial P(Y_T(0) > k|A_T(0))}{\partial \psi_0} P(A_T(0)) + P(Y_T(0) > k|A_T(0)) \frac{\partial P(A_T(0))}{\partial \psi_0} \right. \\
&\quad \left. + \frac{\partial P(Y_T(0) > k|A_T(4))}{\partial \psi_0} P(A_T(4)) + P(Y_T(0) > k|A_T(4)) \frac{\partial P(A_T(4))}{\partial \psi_0} \right\} \\
&\quad - \frac{P(Y_T(0) > k|A_T(0))P(A_T(0)) + P(Y_T(0) > k|A_T(4))P(A_T(4))}{\{P(A_T(0) \cup A_T(4))\}^2} \frac{\partial P(A_T(0) \cup A_T(4))}{\partial \psi_0}, \\
\frac{\partial P(Y_T(1) > k|A_T(0) \cup A_T(4))}{\partial \psi_0} &= \frac{1}{P(A_T(0) \cup A_T(4))} \left\{ \frac{\partial P(Y_T(1) > k|A_T(0))}{\partial \psi_0} P(A_T(0)) + P(Y_T(1) > k|A_T(0)) \frac{\partial P(A_T(0))}{\partial \psi_0} \right. \\
&\quad \left. + \frac{\partial P(Y_T(1) > k|A_T(4))}{\partial \psi_0} P(A_T(4)) + P(Y_T(1) > k|A_T(4)) \frac{\partial P(A_T(4))}{\partial \psi_0} \right\} \\
&\quad - \frac{P(Y_T(1) > k|A_T(0))P(A_T(0)) + P(Y_T(1) > k|A_T(4))P(A_T(4))}{\{P(A_T(0) \cup A_T(4))\}^2} \frac{\partial P(A_T(0) \cup A_T(4))}{\partial \psi_0}, \\
\frac{\partial P(Y_T(0) > k|A_T(0) \cup A_T(4))}{\partial \psi_1} &= \frac{1}{P(A_T(0) \cup A_T(4))} \left\{ \frac{\partial P(Y_T(0) > k|A_T(0))}{\partial \psi_1} P(A_T(0)) + P(Y_T(0) > k|A_T(0)) \frac{\partial P(A_T(0))}{\partial \psi_1} \right. \\
&\quad \left. + \frac{\partial P(Y_T(0) > k|A_T(4))}{\partial \psi_1} P(A_T(4)) + P(Y_T(0) > k|A_T(4)) \frac{\partial P(A_T(4))}{\partial \psi_1} \right\} \\
&\quad - \frac{P(Y_T(0) > k|A_T(0))P(A_T(0)) + P(Y_T(0) > k|A_T(4))P(A_T(4))}{\{P(A_T(0) \cup A_T(4))\}^2} \frac{\partial P(A_T(0) \cup A_T(4))}{\partial \psi_1}, \\
\frac{\partial P(Y_T(1) > k|A_T(0) \cup A_T(4))}{\partial \psi_1} &= \frac{1}{P(A_T(0) \cup A_T(4))} \left\{ \frac{\partial P(Y_T(1) > k|A_T(0))}{\partial \psi_1} P(A_T(0)) + P(Y_T(1) > k|A_T(0)) \frac{\partial P(A_T(0))}{\partial \psi_1} \right. \\
&\quad \left. + \frac{\partial P(Y_T(1) > k|A_T(4))}{\partial \psi_1} P(A_T(4)) + P(Y_T(1) > k|A_T(4)) \frac{\partial P(A_T(4))}{\partial \psi_1} \right\} \\
&\quad - \frac{P(Y_T(1) > k|A_T(0))P(A_T(0)) + P(Y_T(1) > k|A_T(4))P(A_T(4))}{\{P(A_T(0) \cup A_T(4))\}^2} \frac{\partial P(A_T(0) \cup A_T(4))}{\partial \psi_1}, \\
\frac{\partial P(Y_T(0) > k|A_T(0) \cup A_T(4))}{\partial \psi_2} &= \frac{1}{P(A_T(0) \cup A_T(4))} \left\{ \frac{\partial P(Y_T(0) > k|A_T(0))}{\partial \psi_2} P(A_T(0)) + P(Y_T(0) > k|A_T(0)) \frac{\partial P(A_T(0))}{\partial \psi_2} \right. \\
&\quad \left. + \frac{\partial P(Y_T(0) > k|A_T(4))}{\partial \psi_2} P(A_T(4)) + P(Y_T(0) > k|A_T(4)) \frac{\partial P(A_T(4))}{\partial \psi_2} \right\}, \\
\frac{\partial P(Y_T(1) > k|A_T(0) \cup A_T(4))}{\partial \psi_2} &= \frac{1}{P(A_T(0) \cup A_T(4))} \left\{ \frac{\partial P(Y_T(1) > k|A_T(0))}{\partial \psi_2} P(A_T(0)) + P(Y_T(1) > k|A_T(0)) \frac{\partial P(A_T(0))}{\partial \psi_2} \right. \\
&\quad \left. + \frac{\partial P(Y_T(1) > k|A_T(4))}{\partial \psi_2} P(A_T(4)) + P(Y_T(1) > k|A_T(4)) \frac{\partial P(A_T(4))}{\partial \psi_2} \right\}, \\
\frac{\partial P(Y_T(0) > k|A_T(0) \cup A_T(4))}{\partial h_{T,0}(k)} &= \frac{1}{P(A_T(0) \cup A_T(4))} \left\{ \frac{\partial P(Y_T(0) > k|A_T(0))}{\partial h_{T,0}(k)} P(A_T(0)) + \frac{\partial P(Y_T(1) > k|A_T(4))}{\partial h_{T,1}(k)} P(A_T(4)) \right\}, \\
\frac{\partial P(Y_T(1) > k|A_T(0) \cup A_T(4))}{\partial h_{T,1}(k)} &= \frac{1}{P(A_T(0) \cup A_T(4))} \left\{ \frac{\partial P(Y_T(1) > k|A_T(0))}{\partial h_{T,1}(k)} P(A_T(0)) + \frac{\partial P(Y_T(1) > k|A_T(4))}{\partial h_{T,1}(k)} P(A_T(4)) \right\},
\end{aligned}
$$

with

$$
\begin{aligned}
\frac{\partial P(A_T(0))}{\partial \psi_0} &= E_w\left[\frac{\partial g_T(0,w)}{\partial \psi_0}\left\{p_T^*(2,w)-p_T^*(1,w)+1\right\}+g_T(0,w)\left\{\frac{\partial p_T^*(2,w)}{\partial \psi_0}-\frac{\partial p_T^*(1,w)}{\partial \psi_0}\right\}\right],\\
\frac{\partial P(A_T(4))}{\partial \psi_0} &= E_w\left\{\frac{\partial p_T^*(1,w)}{\partial \psi_0}+p_T^*(1,w)\frac{\partial g_T(0,w)}{\partial \psi_0}\right\}-\frac{\partial P(A_T(0))}{\partial \psi_0},\\
\frac{\partial P(A_T(0)\cup A_T(4))}{\partial \psi_0} &= \frac{\partial P(A_T(0))}{\partial \psi_0}+\frac{\partial P(A_T(4))}{\partial \psi_0},\\
\frac{\partial P(A_T(0))}{\partial \psi_1} &= -E_w\left\{g_T(0,w)\frac{\partial p_T^*(1,w)}{\partial \psi_1}\right\},\\
\frac{\partial P(A_T(4))}{\partial \psi_1} &= 2E_w\left\{g_T(0,w)\frac{\partial p_T^*(1,w)}{\partial \psi_1}\right\},\\
\frac{\partial P(A_T(0)\cup A_T(4))}{\partial \psi_1} &= \frac{\partial P(A_T(0))}{\partial \psi_1}+\frac{\partial P(A_T(4))}{\partial \psi_1},\\
\frac{\partial P(A_T(0))}{\partial \psi_2} &= E_w\left\{g_T(0,w)\frac{\partial p_T^*(2,w)}{\partial \psi_2}\right\},\\
\frac{\partial P(A_T(4))}{\partial \psi_2} &= -\frac{\partial P(A_T(0))}{\partial \psi_2}.
\end{aligned}
$$

Detailed Calculations for the Covariance Matrix of $SACE_{k,\cdot}(2,0)$

The log of $\mathrm{SACE}_{k,1}(2,0)$ is

$$
\begin{aligned}
\log \mathrm{SACE}_{k,1}(2,0) \;\; = \;\; & \log P(Y_T(2)>k|A_T(0))-\log\left\{1-P(Y_T(2)>k|A_T(0))\right\}\\
& -\log P(Y_T(0)>k|A_T(0))+\log\left\{1-P(Y_T(0)>k|A_T(0))\right\}.
\end{aligned}
$$

Similar to the calculation of derivatives in $\mathrm{SACE}_{k,1}(1,0)$, the derivatives of $\log \mathrm{SACE}_{k,1}(2,0)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$
\begin{aligned}
\frac{\partial \log SACE_{k,1}(2,0)}{\partial \beta_0(\mathrm{tx})} &= \frac{\partial \log SACE_{k,1}(2,0)}{\partial h_{T,\mathrm{tx}}(k)}\frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta_0(\mathrm{tx})},\\
\frac{\partial \log SACE_{k,1}(2,0)}{\partial \beta(\mathrm{tx})} &= \frac{\partial \log SACE_{k,1}(2,0)}{\partial h_{T,\mathrm{tx}}(k)}\frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta(\mathrm{tx})}.
\end{aligned}
$$

The derivatives of $\log \mathrm{SACE}_{k,1}(2,0)$ with respect to ψ_0, ψ_1, ψ_2, $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$

are, respectively, given by

$$\begin{aligned}
&\frac{\partial \log SACE_{k,1}(2,0)}{\partial \psi_0} \\
=\ & \frac{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial \psi_0}}{P(Y_T(2)>k|A_T(0))\{1-P(Y_T(2)>k|A_T(0))\}} - \frac{\frac{\partial P(Y_T(0)>k|A_T(0))}{\partial \psi_0}}{P(Y_T(0)>k|A_T(0))\{1-P(Y_T(0)>k|A_T(0))\}}, \quad (6) \\
&\frac{\partial \log SACE_{k,1}(2,0)}{\partial \psi_1} \\
=\ & \frac{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial \psi_1}}{P(Y_T(2)>k|A_T(0))\{1-P(Y_T(2)>k|A_T(0))\}} - \frac{\frac{\partial P(Y_T(0)>k|A_T(0))}{\partial \psi_1}}{P(Y_T(0)>k|A_T(0))\{1-P(Y_T(0)>k|A_T(0))\}}, \quad (7) \\
&\frac{\partial \log SACE_{k,1}(2,0)}{\partial \psi_2} \\
=\ & \frac{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial \psi_2}}{P(Y_T(2)>k|A_T(0))\{1-P(Y_T(2)>k|A_T(0))\}} - \frac{\frac{\partial P(Y_T(0)>k|A_T(0))}{\partial \psi_2}}{P(Y_T(0)>k|A_T(0))\{1-P(Y_T(0)>k|A_T(0))\}}, \quad (8) \\
&\frac{\partial \log SACE_{k,1}(2,0)}{\partial h_{T,0}(k)} = -\frac{\frac{\partial P(Y_T(0)>k|A_T(0)}{\partial h_{T,0}(k)}}{P(Y_T(0)>k|A_T(0))\{1-P(Y_T(0)>k|A_T(0))\}}, \quad (9) \\
&\frac{\partial \log SACE_{k,1}(2,0)}{\partial h_{T,1}(k)} = 0, \\
&\frac{\partial \log SACE_{k,1}(2,0)}{\partial h_{T,2}(k)} = \frac{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial h_{T,2}(k)}}{P(Y_T(2)>k|A_T(0))\{1-P(Y_T(2)>k|A_T(0))\}}. \quad (10)
\end{aligned}$$

where the numerator parts in (6)-(10) are calculated using numerical derivatives. We now describe the calculation for the covariance matrix for the log of $SACE_{k,2}(2,0)$. The log of $SACE_{k,2}(2,0)$ is

$$\begin{aligned}
\log \mathrm{SACE}_{k,2}(2,0) \;=\; & \log P(Y_T(2)>k|A_T(5)) - \log\{1-P(Y_T(2)>k|A_T(5))\} \\
& - \log P(Y_T(0)>k|A_T(5)) + \log\{1-P(Y_T(0)>k|A_T(5))\}.
\end{aligned}$$

The derivatives of $\log \mathrm{SACE}_{k,2}(2,0)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\begin{aligned}
\frac{\partial \log SACE_{k,2}(2,0)}{\partial \beta_0(\mathrm{tx})} &= \frac{\partial \log SACE_{k,2}(2,0)}{\partial h_{T,\mathrm{tx}}(k)} \frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta_0(\mathrm{tx})}, \\
\frac{\partial \log SACE_{k,2}(2,0)}{\partial \beta(\mathrm{tx})} &= \frac{\partial \log SACE_{k,2}(2,0)}{\partial h_{T,\mathrm{tx}}(k)} \frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta(\mathrm{tx})}.
\end{aligned}$$

The derivatives of $\log \mathrm{SACE}_{k,2}(2,0)$ with respect to ψ_0, ψ_1, ψ_2, $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$

are given by

$$\begin{aligned}
\frac{\partial \log SACE_{k,2}(2,0)}{\partial \psi_0} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(5))}{\partial \psi_0}}{P(Y_T(2)>k|A_T(5))\left\{1-P(Y_T(2)>k|A_T(5))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(5))}{\partial \psi_0}}{P(Y_T(0)>k|A_T(5))\left\{1-P(Y_T(0)>k|A_T(5))\right\}}, \\
\frac{\partial \log SACE_{k,2}(2,0)}{\partial \psi_1} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(5))}{\partial \psi_1}}{P(Y_T(2)>k|A_T(5))\left\{1-P(Y_T(2)>k|A_T(5))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(5))}{\partial \psi_1}}{P(Y_T(0)>k|A_T(5))\left\{1-P(Y_T(0)>k|A_T(5))\right\}}, \\
\frac{\partial \log SACE_{k,2}(2,0)}{\partial \psi_2} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(5))}{\partial \psi_2}}{P(Y_T(2)>k|A_T(5))\left\{1-P(Y_T(2)>k|A_T(5))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(5))}{\partial \psi_2}}{P(Y_T(0)>k|A_T(5))\left\{1-P(Y_T(0)>k|A_T(5))\right\}}, \\
\frac{\partial \log SACE_{k,2}(2,0)}{\partial h_{T,0}(k)} &= -\frac{\frac{\partial P(Y_T(0)>k|A_T(5))}{\partial h_{T,0}(k)}}{P(Y_T(0)>k|A_T(5))\left\{1-P(Y_T(0)>k|A_T(5))\right\}}, \\
\frac{\partial \log SACE_{k,2}(2,0)}{\partial h_{T,1}(k)} &= 0, \\
\frac{\partial \log SACE_{k,2}(2,0)}{\partial h_{T,2}(k)} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(5))}{\partial h_{T,2}(k)}}{P(Y_T(2)>k|A_T(5))\left\{1-P(Y_T(2)>k|A_T(5))\right\}},
\end{aligned}$$

where

$$\begin{aligned}
\frac{\partial P(Y_T(j)>k|A_T(5))}{\partial \psi_{\text{tx}}} &= \frac{\tau \frac{\partial P(Y_T(j)>k|A_T(0))}{\partial \psi_{\text{tx}}}}{\left\{1+(\tau-1)P(Y_T(j)>k|A_T(0))\right\}^2}, \\
\frac{\partial P(Y_T(0)>k|A_T(5))}{\partial h_{T,0}(k)} &= \frac{\tau \frac{\partial P(Y_T(0)>k|A_T(0))}{\partial h_{T,0}(k)}}{\left\{1+(\tau-1)P(Y_T(0)>k|A_T(0))\right\}^2}, \\
\frac{\partial P(Y_T(2)>k|A_T(5))}{\partial h_{T,2}(k)} &= \frac{\tau \frac{\partial P(Y_T(2)>k|A_T(0))}{\partial h_{T,2}(k)}}{\left\{1+(\tau-1)P(Y_T(2)>k|A_T(0))\right\}^2},
\end{aligned}$$

for $j = 0, 2$ and $\text{tx} = 0, 1, 2$.

We now describe the calculation for the covariance matrix for the log of $SACE_{k,3}(2,0)$.

The log of $SACE_{k,3}(2,0)$ is

$$\begin{aligned}
\log \text{SACE}_{k,3}(2,0) &= \log P(Y_T(2)>k|A_T(0)\cup A_T(5)) - \log\left\{1-P(Y_T(2)>k|A_T(0)\cup A_T(5))\right\} \\
&\quad - \log P(Y_T(0)>k|A_T(0)\cup A_T(5)) + \log\left\{1-P(Y_T(0)>k|A_T(0)\cup A_T(5))\right\}.
\end{aligned}$$

The derivatives of $\log \mathrm{SACE}_{k,3}(2,0)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\frac{\partial \log SACE_{k,3}(2,0)}{\partial \beta_0(\mathrm{tx})} = \frac{\partial \log SACE_{k,3}(2,0)}{\partial h_{T,\mathrm{tx}}(k)} \frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta_0(\mathrm{tx})},$$
$$\frac{\partial \log SACE_{k,3}(2,0)}{\partial \beta(\mathrm{tx})} = \frac{\partial \log SACE_{k,3}(2,0)}{\partial h_{T,\mathrm{tx}}(k)} \frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta(\mathrm{tx})}.$$

The derivatives of $\log \mathrm{SACE}_{k,3}(2,0)$ with respect to ψ_0, ψ_1, ψ_2, $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$ are given by

$$\begin{aligned}
\frac{\partial \log SACE_{k,3}(2,0)}{\partial \psi_0} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(5))}{\partial \psi_0}}{P(Y_T(2) > k|A_T(0) \cup A_T(5)) \{1 - P(Y_T(2) > k|A_T(0) \cup A_T(5))\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(0)\cup A_T(5))}{\partial \psi_0}}{P(Y_T(0) > k|A_T(0) \cup A_T(5)) \{1 - P(Y_T(0) > k|A_T(0) \cup A_T(5))\}}, \\
\frac{\partial \log SACE_{k,3}(2,0)}{\partial \psi_1} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(5))}{\partial \psi_1}}{P(Y_T(2) > k|A_T(0) \cup A_T(5)) \{1 - P(Y_T(1) > k|A_T(0) \cup A_T(5))\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(0)\cup A_T(5))}{\partial \psi_1}}{P(Y_T(0) > k|A_T(0) \cup A_T(5)) \{1 - P(Y_T(0) > k|A_T(0) \cup A_T(5))\}}, \\
\frac{\partial \log SACE_{k,3}(2,0)}{\partial \psi_2} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(5))}{\partial \psi_2}}{P(Y_T(2) > k|A_T(0) \cup A_T(5)) \{1 - P(Y_T(2) > k|A_T(0) \cup A_T(5))\}} \\
&\quad - \frac{\frac{\partial P(Y_T(0)>k|A_T(0)\cup A_T(5))}{\partial \psi_2}}{P(Y_T(0) > k|A_T(0) \cup A_T(5)) \{1 - P(Y_T(0) > k|A_T(0) \cup A_T(5))\}}, \\
\frac{\partial \log SACE_{k,3}(2,0)}{\partial h_{T,0}(k)} &= - \frac{\frac{\partial P(Y_T(0)>k|A_T(0)\cup A_T(5))}{\partial h_{T,0}(k)}}{P(Y_T(0) > k|A_T(0) \cup A_T(5)) \{1 - P(Y_T(0) > k|A_T(0) \cup A_T(5))\}}, \\
\frac{\partial \log SACE_{k,3}(2,0)}{\partial h_{T,1}(k)} &= 0, \\
\frac{\partial \log SACE_{k,3}(2,0)}{\partial h_{T,2}(k)} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(5))}{\partial h_{T,2}(k)}}{P(Y_T(2) > k|A_T(0) \cup A_T(5)) \{1 - P(Y_T(2) > k|A_T(0) \cup A_T(5))\}},
\end{aligned}$$

where

$$\frac{\partial P(Y_T(0)>k|A_T(0)\cup A_T(5))}{\partial\psi_0} = \frac{1}{P(A_T(0)\cup A_T(5))}\left\{\frac{\partial P(Y_T(0)>k|A_T(0))}{\partial\psi_0}P(A_T(0)) + P(Y_T(0)>k|A_T(0))\frac{\partial P(A_T(0))}{\partial\psi_0}\right.$$
$$\left.+\frac{\partial P(Y_T(0)>k|A_T(5))}{\partial\psi_0}P(A_T(5)) + P(Y_T(0)>k|A_T(5))\frac{\partial P(A_T(5))}{\partial\psi_0}\right\}$$
$$-\frac{P(Y_T(0)>k|A_T(0))P(A_T(0)) + P(Y_T(0)>k|A_T(5))P(A_T(5))}{\{P(A_T(0)\cup A_T(5))\}^2}\frac{\partial P(A_T(0)\cup A_T(5))}{\partial\psi_0},$$
$$\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(5))}{\partial\psi_0} = \frac{1}{P(A_T(0)\cup A_T(5))}\left\{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial\psi_0}P(A_T(0)) + P(Y_T(2)>k|A_T(0))\frac{\partial P(A_T(0))}{\partial\psi_0}\right.$$
$$\left.+\frac{\partial P(Y_T(2)>k|A_T(5))}{\partial\psi_0}P(A_T(5)) + P(Y_T(2)>k|A_T(5))\frac{\partial P(A_T(5))}{\partial\psi_0}\right\}$$
$$-\frac{P(Y_T(2)>k|A_T(0))P(A_T(0)) + P(Y_T(2)>k|A_T(5))P(A_T(5))}{\{P(A_T(0)\cup A_T(5))\}^2}\frac{\partial P(A_T(0)\cup A_T(5))}{\partial\psi_0},$$
$$\frac{\partial P(Y_T(0)>k|A_T(0)\cup A_T(5))}{\partial\psi_1} = \frac{1}{P(A_T(0)\cup A_T(5))}\left\{\frac{\partial P(Y_T(0)>k|A_T(0))}{\partial\psi_1}P(A_T(0)) + P(Y_T(0)>k|A_T(0))\frac{\partial P(A_T(0))}{\partial\psi_1}\right.$$
$$\left.+\frac{\partial P(Y_T(0)>k|A_T(5))}{\partial\psi_1}P(A_T(5)) + P(Y_T(0)>k|A_T(5))\frac{\partial P(A_T(5))}{\partial\psi_1}\right\},$$
$$\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(5))}{\partial\psi_1} = \frac{1}{P(A_T(0)\cup A_T(5))}\left\{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial\psi_1}P(A_T(0)) + P(Y_T(2)>k|A_T(0))\frac{\partial P(A_T(0))}{\partial\psi_1}\right.$$
$$\left.+\frac{\partial P(Y_T(2)>k|A_T(5))}{\partial\psi_1}P(A_T(5)) + P(Y_T(2)>k|A_T(5))\frac{\partial P(A_T(5))}{\partial\psi_1}\right\},$$
$$\frac{\partial P(Y_T(0)>k|A_T(0)\cup A_T(5))}{\partial\psi_2} = \frac{1}{P(A_T(0)\cup A_T(5))}\left\{\frac{\partial P(Y_T(0)>k|A_T(0))}{\partial\psi_2}P(A_T(0)) + P(Y_T(0)>k|A_T(0))\frac{\partial P(A_T(0))}{\partial\psi_2}\right.$$
$$\left.+\frac{\partial P(Y_T(0)>k|A_T(5))}{\partial\psi_2}P(A_T(5))\right\}$$
$$-\frac{P(Y_T(0)>k|A_T(0))P(A_T(0)) + P(Y_T(0)>k|A_T(5))P(A_T(5))}{\{P(A_T(0)\cup A_T(5))\}^2}\frac{\partial P(A_T(0)\cup A_T(5))}{\partial\psi_2},$$
$$\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(5))}{\partial\psi_2} = \frac{1}{P(A_T(0)\cup A_T(5))}\left\{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial\psi_2}P(A_T(0)) + P(Y_T(2)>k|A_T(0))\frac{\partial P(A_T(0))}{\partial\psi_2}\right.$$
$$\left.+\frac{\partial P(Y_T(2)>k|A_T(5))}{\partial\psi_2}P(A_T(5))\right\}$$
$$-\frac{P(Y_T(2)>k|A_T(0))P(A_T(0)) + P(Y_T(2)>k|A_T(5))P(A_T(5))}{\{P(A_T(0)\cup A_T(5))\}^2}\frac{\partial P(A_T(0)\cup A_T(5))}{\partial\psi_2},$$
$$\frac{\partial P(Y_T(0)>k|A_T(0)\cup A_T(5))}{\partial h_{T,0}} = \frac{1}{P(A_T(0)\cup A_T(5))}\left\{\frac{\partial P(Y_T(0)>k|A_T(0))}{\partial h_{T,0}(k)}P(A_T(0)) + \frac{\partial P(Y_T(0)>k|A_T(5))}{\partial h_{T,0}(k)}P(A_T(5))\right\},$$
$$\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(5))}{\partial h_{T,2}} = \frac{1}{P(A_T(0)\cup A_T(5))}\left\{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial h_{T,2}(k)}P(A_T(0)) + \frac{\partial P(Y_T(2)>k|A_T(5))}{\partial h_{T,2}(k)}P(A_T(5))\right\},$$

with

$$\frac{\partial P(A_T(5))}{\partial\psi_0} = E_w\left\{\frac{\partial g_T(0,w)}{\partial\psi_0}(1-p_T^*(1,w)) - g_T^*(0,w)\frac{\partial p_T^*(1,w)}{\partial\psi_0}\right\},$$
$$\frac{\partial P(A_T(0)\cup A_T(5))}{\partial\psi_0} = \frac{\partial P(A_T(0))}{\partial\psi_0} + \frac{\partial P(A_T(5))}{\partial\psi_0},$$
$$\frac{\partial P(A_T(5))}{\partial\psi_1} = -E_w\left\{g_T(0,w)\frac{\partial p_T^*(1,w)}{\partial\psi_1}\right\},$$
$$\frac{\partial P(A_T(0)\cup A_T(5))}{\partial\psi_2} = \frac{\partial P(A_T(0))}{\partial\psi_2}.$$

Detailed Calculations for the Covariance Matrix of $SACE_{k,\cdot}(2,1)$
The log of $\text{SACE}_{k,1}(2,1)$ is

$$\begin{aligned}\log \text{SACE}_{k,1}(2,1) &= \log P(Y_T(2) > k|A_T(0)) - \log\{1 - P(Y_T(2) > k|A_T(0))\} \\ &\quad - \log P(Y_T(1) > k|A_T(0)) + \log\{1 - P(Y_T(1) > k|A_T(0))\}.\end{aligned}$$

The derivatives of $\log \text{SACE}_{k,1}(2,1)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\begin{aligned}\frac{\partial \log SACE_{k,1}(2,1)}{\partial \beta_0(\text{tx})} &= \frac{\partial \log SACE_{k,1}(2,1)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta_0(\text{tx})}, \\ \frac{\partial \log SACE_{k,1}(2,1)}{\partial \beta(\text{tx})} &= \frac{\partial \log SACE_{k,1}(2,1)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta(\text{tx})}.\end{aligned}$$

The derivatives of $\log \text{SACE}_{k,1}(2,1)$ with respect to ψ_0, ψ_1, ψ_2, $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$ are given by

$$\begin{aligned}&\frac{\partial \log SACE_{k,1}(2,1)}{\partial \psi_0} \\ &= \frac{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial \psi_0}}{P(Y_T(2) > k|A_T(0))\{1 - P(Y_T(2) > k|A_T(0))\}} - \frac{\frac{\partial P(Y_T(1)>k|A_T(0))}{\partial \psi_0}}{P(Y_T(1) > k|A_T(0))\{1 - P(Y_T(1) > k|A_T(0))\}}, \\ &\frac{\partial \log SACE_{k,1}(2,1)}{\partial \psi_1} \\ &= \frac{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial \psi_1}}{P(Y_T(2) > k|A_T(0))\{1 - P(Y_T(2) > k|A_T(0))\}} - \frac{\frac{\partial P(Y_T(1)>k|A_T(0))}{\partial \psi_1}}{P(Y_T(1) > k|A_T(0))\{1 - P(Y_T(1) > k|A_T(0))\}}, \\ &\frac{\partial \log SACE_{k,1}(2,1)}{\partial \psi_2} \\ &= \frac{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial \psi_2}}{P(Y_T(2) > k|A_T(0))\{1 - P(Y_T(2) > k|A_T(0))\}} - \frac{\frac{\partial P(Y_T(1)>k|A_T(0))}{\partial \psi_2}}{P(Y_T(1) > k|A_T(0))\{1 - P(Y_T(1) > k|A_T(0))\}}, \\ &\frac{\partial \log SACE_{k,1}(2,1)}{\partial h_{T,0}(k)} = 0, \\ &\frac{\partial \log SACE_{k,1}(2,1)}{\partial h_{T,1}(k)} = -\frac{\frac{\partial P(Y_T(1)>k|A_T(0)}{\partial h_{T,1}(k)}}{P(Y_T(1) > k|A_T(0))\{1 - P(Y_T(1) > k|A_T(0))\}}, \\ &\frac{\partial \log SACE_{k,1}(2,1)}{\partial h_{T,2}(k)} = \frac{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial h_{T,2}(k)}}{P(Y_T(2) > k|A_T(0))\{1 - P(Y_T(2) > k|A_T(0))\}}.\end{aligned}$$

We now describe the calculation for the covariance matrix for the log of $SACE_{k,2}(2,1)$. The log of $SACE_{k,2}(2,1)$ is

$$\begin{aligned}\log \text{SACE}_{k,2}(2,1) &= \log P(Y_T(2) > k|A_T(1)) - \log\{1 - P(Y_T(2) > k|A_T(1))\} \\ &\quad - \log P(Y_T(1) > k|A_T(1)) + \log\{1 - P(Y_T(1) > k|A_T(1))\}.\end{aligned}$$

The derivatives of $\log \mathrm{SACE}_{k,2}(2,1)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\frac{\partial \log SACE_{k,2}(2,1)}{\partial \beta_0(\mathrm{tx})} = \frac{\partial \log SACE_{k,2}(2,1)}{\partial h_{T,\mathrm{tx}}(k)} \frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta_0(\mathrm{tx})},$$
$$\frac{\partial \log SACE_{k,2}(2,1)}{\partial \beta(\mathrm{tx})} = \frac{\partial \log SACE_{k,2}(2,1)}{\partial h_{T,\mathrm{tx}}(k)} \frac{\partial h_{T,\mathrm{tx}}(k)}{\partial \beta(\mathrm{tx})}.$$

The derivatives of $\log \mathrm{SACE}_{k,2}(2,1)$ with respect to ψ_0, ψ_1, ψ_2, $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$ are given by

$$\begin{aligned}
\frac{\partial \log SACE_{k,2}(2,1)}{\partial \psi_0} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(1))}{\partial \psi_0}}{P(Y_T(2)>k|A_T(1))\left\{1-P(Y_T(2)>k|A_T(1))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(1)>k|A_T(1))}{\partial \psi_0}}{P(Y_T(1)>k|A_T(1))\left\{1-P(Y_T(1)>k|A_T(1))\right\}}, \\
\frac{\partial \log SACE_{k,2}(2,1)}{\partial \psi_1} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(1))}{\partial \psi_1}}{P(Y_T(2)>k|A_T(1))\left\{1-P(Y_T(2)>k|A_T(1))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(1)>k|A_T(1))}{\partial \psi_1}}{P(Y_T(1)>k|A_T(1))\left\{1-P(Y_T(1)>k|A_T(1))\right\}}, \\
\frac{\partial \log SACE_{k,2}(2,1)}{\partial \psi_2} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(1))}{\partial \psi_2}}{P(Y_T(2)>k|A_T(1))\left\{1-P(Y_T(2)>k|A_T(1))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(1)>k|A_T(1))}{\partial \psi_2}}{P(Y_T(1)>k|A_T(1))\left\{1-P(Y_T(1)>k|A_T(1))\right\}}, \\
\frac{\partial \log SACE_{k,2}(2,1)}{\partial h_{T,0}(k)} &= 0, \\
\frac{\partial \log SACE_{k,2}(2,1)}{\partial h_{T,1}(k)} &= -\frac{\frac{\partial P(Y_T(1)>k|A_T(1))}{\partial h_{T,1}(k)}}{P(Y_T(1)>k|A_T(1))\left\{1-P(Y_T(1)>k|A_T(5))\right\}}, \\
\frac{\partial \log SACE_{k,2}(2,1)}{\partial h_{T,2}(k)} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(1))}{\partial h_{T,2}(k)}}{P(Y_T(2)>k|A_T(1))\left\{1-P(Y_T(2)>k|A_T(1))\right\}},
\end{aligned}$$

where

$$\frac{\partial P(Y_T(j)>k|A_T(1))}{\partial \psi_{\text{tx}}} = \frac{\tau \frac{\partial P(Y_T(j)>k|A_T(0))}{\partial \psi_{\text{tx}}}}{\{1+(\tau-1)P(Y_T(j)>k|A_T(0))\}^2},$$

$$\frac{\partial P(Y_T(1)>k|A_T(1))}{\partial h_{T,1}(k)} = \frac{\tau \frac{\partial P(Y_T(1)>k|A_T(0))}{\partial h_{T,1}(k)}}{\{1+(\tau-1)P(Y_T(1)>k|A_T(0))\}^2},$$

$$\frac{\partial P(Y_T(2)>k|A_T(1))}{\partial h_{T,2}(k)} = \frac{\tau \frac{\partial P(Y_T(2)>k|A_T(0))}{\partial h_{T,2}(k)}}{\{1+(\tau-1)P(Y_T(2)>k|A_T(0))\}^2},$$

for $j=1,2$ and $\text{tx}=0,1,2$.
We finally describe the calculation for the covariance matrix for the log of $SACE_{k,3}(2,1)$. The log of $SACE_{k,3}(2,1)$ is

$$\begin{aligned}\log \text{SACE}_{k,3}(2,1) &= \log P(Y_T(2)>k|A_T(0)\cup A_T(1)) - \log\{1-P(Y_T(2)>k|A_T(0)\cup A_T(1))\} \\ &\quad - \log P(Y_T(1)>k|A_T(0)\cup A_T(1)) + \log\{1-P(Y_T(1)>k|A_T(0)\cup A_T(1))\}.\end{aligned}$$

The derivatives of $\log \text{SACE}_{k,3}(2,1)$ with respect to $\beta_0(0)$, $\beta_0(1)$, $\beta_0(2)$, $\beta(0)$, $\beta(1)$, and $\beta(2)$ are, respectively, given by

$$\frac{\partial \log SACE_{k,3}(2,1)}{\partial \beta_0(\text{tx})} = \frac{\partial \log SACE_{k,3}(2,1)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta_0(\text{tx})},$$

$$\frac{\partial \log SACE_{k,3}(2,1)}{\partial \beta(\text{tx})} = \frac{\partial \log SACE_{k,3}(2,1)}{\partial h_{T,\text{tx}}(k)} \frac{\partial h_{T,\text{tx}}(k)}{\partial \beta(\text{tx})}.$$

The derivatives of $\log \text{SACE}_{k,3}(2,1)$ with respect to ψ_0, ψ_1, ψ_2, $h_{T,0}(k)$, $h_{T,1}(k)$, and $h_{T,2}(k)$

are given by

$$
\begin{aligned}
\frac{\partial \log SACE_{k,3}(2,1)}{\partial \psi_0} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(1))}{\partial \psi_0}}{P(Y_T(2)>k|A_T(0)\cup A_T(1))\left\{1-P(Y_T(2)>k|A_T(0)\cup A_T(1))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(1)>k|A_T(0)\cup A_T(1))}{\partial \psi_0}}{P(Y_T(1)>k|A_T(0)\cup A_T(1))\left\{1-P(Y_T(1)>k|A_T(0)\cup A_T(1))\right\}}, \\
\frac{\partial \log SACE_{k,3}(2,1)}{\partial \psi_1} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(1))}{\partial \psi_1}}{P(Y_T(2)>k|A_T(0)\cup A_T(1))\left\{1-P(Y_T(2)>k|A_T(0)\cup A_T(1))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(1)>k|A_T(0)\cup A_T(1))}{\partial \psi_1}}{P(Y_T(1)>k|A_T(0)\cup A_T(1))\left\{1-P(Y_T(1)>k|A_T(0)\cup A_T(1))\right\}}, \\
\frac{\partial \log SACE_{k,3}(2,1)}{\partial \psi_2} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(1))}{\partial \psi_2}}{P(Y_T(2)>k|A_T(0)\cup A_T(1))\left\{1-P(Y_T(2)>k|A_T(0)\cup A_T(1))\right\}} \\
&\quad - \frac{\frac{\partial P(Y_T(1)>k|A_T(0)\cup A_T(1))}{\partial \psi_2}}{P(Y_T(1)>k|A_T(0)\cup A_T(1))\left\{1-P(Y_T(1)>k|A_T(0)\cup A_T(1))\right\}}, \\
\frac{\partial \log SACE_{k,3}(2,1)}{\partial h_{T,0}(k)} &= 0, \\
\frac{\partial \log SACE_{k,3}(2,1)}{\partial h_{T,1}(k)} &= -\frac{\frac{\partial P(Y_T(1)>k|A_T(0)\cup A_T(1))}{\partial h_{T,1}(k)}}{P(Y_T(1)>k|A_T(0)\cup A_T(1))\left\{1-P(Y_T(1)>k|A_T(0)\cup A_T(1))\right\}}, \\
\frac{\partial \log SACE_{k,3}(2,1)}{\partial h_{T,2}(k)} &= \frac{\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(1))}{\partial h_{T,2}(k)}}{P(Y_T(2)>k|A_T(0)\cup A_T(1))\left\{1-P(Y_T(2)>k|A_T(0)\cup A_T(1))\right\}},
\end{aligned}
$$

where

$$\frac{\partial P(Y_T(1)>k|A_T(0)\cup A_T(1))}{\partial\psi_0} = \frac{1}{P(A_T(0)\cup A_T(1))}\left\{\frac{\partial P(Y_T(1)>k|A_T(0))}{\partial\psi_0}P(A_T(0)) + P(Y_T(1)>k|A_T(0))\frac{\partial P(A_T(0))}{\partial\psi_0}\right.$$
$$\left.+\frac{\partial P(Y_T(1)>k|A_T(1))}{\partial\psi_0}P(A_T(1)) + P(Y_T(1)>k|A_T(1))\frac{\partial P(A_T(1))}{\partial\psi_0}\right\}$$
$$-\frac{P(Y_T(1)>k|A_T(0))P(A_T(0)) + P(Y_T(1)>k|A_T(1))P(A_T(1))}{\{P(A_T(0)\cup A_T(1))\}^2}\frac{\partial P(A_T(0)\cup A_T(1))}{\partial\psi_0},$$

$$\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(1))}{\partial\psi_0} = \frac{1}{P(A_T(0)\cup A_T(1))}\left\{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial\psi_0}P(A_T(0)) + P(Y_T(2)>k|A_T(0))\frac{\partial P(A_T(0))}{\partial\psi_0}\right.$$
$$\left.+\frac{\partial P(Y_T(2)>k|A_T(1))}{\partial\psi_0}P(A_T(1)) + P(Y_T(2)>k|A_T(1))\frac{\partial P(A_T(1))}{\partial\psi_0}\right\}$$
$$-\frac{P(Y_T(2)>k|A_T(0))P(A_T(0)) + P(Y_T(2)>k|A_T(1))P(A_T(1))}{\{P(A_T(0)\cup A_T(1))\}^2}\frac{\partial P(A_T(0)\cup A_T(1))}{\partial\psi_0},$$

$$\frac{\partial P(Y_T(1)>k|A_T(0)\cup A_T(1))}{\partial\psi_1} = \frac{1}{P(A_T(0)\cup A_T(1))}\left\{\frac{\partial P(Y_T(1)>k|A_T(0))}{\partial\psi_1}P(A_T(0)) + P(Y_T(1)>k|A_T(0))\frac{\partial P(A_T(0))}{\partial\psi_1}\right.$$
$$\left.+\frac{\partial P(Y_T(1)>k|A_T(1))}{\partial\psi_1}P(A_T(1)) + P(Y_T(1)>k|A_T(1))\frac{\partial P(A_T(1))}{\partial\psi_1}\right\}$$
$$-\frac{P(Y_T(1)>k|A_T(0))P(A_T(0)) + P(Y_T(1)>k|A_T(1))P(A_T(1))}{\{P(A_T(0)\cup A_T(1))\}^2}\frac{\partial P(A_T(0)\cup A_T(1))}{\partial\psi_1},$$

$$\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(1))}{\partial\psi_1} = \frac{1}{P(A_T(0)\cup A_T(1))}\left\{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial\psi_1}P(A_T(0)) + P(Y_T(2)>k|A_T(0))\frac{\partial P(A_T(0))}{\partial\psi_1}\right.$$
$$\left.+\frac{\partial P(Y_T(2)>k|A_T(1))}{\partial\psi_1}P(A_T(1)) + P(Y_T(2)>k|A_T(1))\frac{\partial P(A_T(1))}{\partial\psi_1}\right\}$$
$$-\frac{P(Y_T(2)>k|A_T(0))P(A_T(0)) + P(Y_T(2)>k|A_T(1))P(A_T(1))}{\{P(A_T(0)\cup A_T(1))\}^2}\frac{\partial P(A_T(0)\cup A_T(1))}{\partial\psi_1},$$

$$\frac{\partial P(Y_T(1)>k|A_T(0)\cup A_T(1))}{\partial\psi_2} = \frac{1}{P(A_T(0)\cup A_T(1))}\left\{\frac{\partial P(Y_T(1)>k|A_T(0))}{\partial\psi_2}P(A_T(0)) + P(Y_T(1)>k|A_T(0))\frac{\partial P(A_T(0))}{\partial\psi_2}\right.$$
$$\left.+\frac{\partial P(Y_T(1)>k|A_T(1))}{\partial\psi_2}P(A_T(1)) + P(Y_T(1)>k|A_T(1))\frac{\partial P(A_T(1))}{\partial\psi_2}\right\}$$
$$-\frac{P(Y_T(1)>k|A_T(0))P(A_T(0)) + P(Y_T(1)>k|A_T(1))P(A_T(1))}{\{P(A_T(0)\cup A_T(1))\}^2}\frac{\partial P(A_T(0)\cup A_T(1))}{\partial\psi_2},$$

$$\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(1))}{\partial\psi_2} = \frac{1}{P(A_T(0)\cup A_T(1))}\left\{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial\psi_2}P(A_T(0)) + P(Y_T(2)>k|A_T(0))\frac{\partial P(A_T(0))}{\partial\psi_2}\right.$$
$$\left.+\frac{\partial P(Y_T(2)>k|A_T(1))}{\partial\psi_2}P(A_T(1)) + P(Y_T(2)>k|A_T(1))\frac{\partial P(A_T(1))}{\partial\psi_2}\right\}$$
$$-\frac{P(Y_T(2)>k|A_T(0))P(A_T(0)) + P(Y_T(2)>k|A_T(1))P(A_T(1))}{\{P(A_T(0)\cup A_T(1))\}^2}\frac{\partial P(A_T(0)\cup A_T(1))}{\partial\psi_2},$$

$$\frac{\partial P(Y_T(1)>k|A_T(0)\cup A_T(1))}{\partial h_{T,1}(k)} = \frac{1}{P(A_T(0)\cup A_T(1))}\left\{\frac{\partial P(Y_T(1)>k|A_T(0))}{\partial h_{T,1}(k)}P(A_T(0)) + \frac{\partial P(Y_T(1)>k|A_T(1))}{\partial h_{T,1}(k)}P(A_T(1))\right\},$$
$$\frac{\partial P(Y_T(2)>k|A_T(0)\cup A_T(1))}{\partial h_{T,2}(k)} = \frac{1}{P(A_T(0)\cup A_T(1))}\left\{\frac{\partial P(Y_T(2)>k|A_T(0))}{\partial h_{T,2}(k)}P(A_T(0)) + \frac{\partial P(Y_T(2)>k|A_T(1))}{\partial h_{T,2}(k)}P(A_T(1))\right\},$$

with

$$
\begin{aligned}
\frac{\partial P(A_T(1))}{\partial \psi_0} &= -E_w \left\{ \frac{\partial g_T(0,w)}{\partial \psi_0} p_T^*(2,w) + g_T(0,w) \frac{\partial p_T^*(2,w)}{\partial \psi_0} \right\} - \frac{\partial P(A_T(2))}{\partial \psi_0}, \\
\frac{\partial P(A_T(2))}{\partial \psi_0} &= E_w \left[-\frac{\partial q_T^*(w)}{\partial \psi_0} \{1 - g_T(0,w) - g_T(1,w) + p_T^*(1,w) g_T(0,w)\} \right. \\
&\quad \left. + (1 - q_T^*(w)) \left\{ -\frac{\partial g_T(0,w)}{\partial \psi_0} + \frac{\partial p_T^*(1,w)}{\partial \psi_0} g_T(0,w) + p_T^*(1,w) \frac{\partial g_T(0,w)}{\partial \psi_0} \right\} \right], \\
\frac{\partial P(A_T(0) \cup A_T(1))}{\partial \psi_0} &= \frac{\partial P(A_T(0))}{\partial \psi_0} + \frac{\partial P(A_T(1))}{\partial \psi_0}, \\
\frac{\partial P(A_T(1))}{\partial \psi_1} &= -E_w \left[\frac{\partial q_T^*(w)}{\partial \psi_1} \{1 - g_T(0,w) - g_T(1,w) + p_T^*(1,w) g_T(0,w)\} \right. \\
&\quad \left. + (1 - q_T^*(w)) \left\{ \frac{\partial g_T(1,w)}{\partial \psi_1} - g_T(0,w) \frac{\partial p_T^*(1,w)}{\partial \psi_1} \right\} \right], \\
\frac{\partial P(A_T(0) \cup A_T(1))}{\partial \psi_1} &= \frac{\partial P(A_T(0))}{\partial \psi_1} + \frac{\partial P(A_T(1))}{\partial \psi_1}, \\
\frac{\partial P(A_T(1))}{\partial \psi_2} &= E_w \left\{ \frac{\partial g_T(2,w)}{\partial \psi_2} - g_T(0,w) \frac{\partial p_T^*(2,w)}{\partial \psi_2} \right\} - \frac{\partial P(A_T(2))}{\partial \psi_2}, \\
\frac{\partial P(A_T(2))}{\partial \psi_2} &= -E_w \left[\frac{\partial q_T^*(w)}{\partial \psi_2} \{1 - g_T(0,w) - g_T(1,w) + p_T^*(1,w) g_T(0,w)\} \right], \\
\frac{\partial P(A_T(0) \cup A_T(1))}{\partial \psi_2} &= \frac{\partial P(A_T(0))}{\partial \psi_2} + \frac{\partial P(A_T(1))}{\partial \psi_2},
\end{aligned}
$$

Numerical derivatives

Numerical derivatives are computed as local slopes based on small perturbations of parameter estimates (Press et al., 1992). Let $f(\omega)$ be the left hand side of (25), (26), or (32) where $\omega = (\psi_0, \psi_1, \psi_2, h_{T,1}(k), h_{T,2}(k))$, and let $\omega^{(-i)}$ be a vector with the ith element of ω removed. The ith element of ν, $\omega^{(i)}$, is perturbed as follows:

$$
\begin{aligned}
\delta_i &= c_1 \left(\frac{\omega^{(i)}}{\omega^{(i)} + c_2} \right), \\
\omega_1^{(i)} &= \begin{cases} \omega^{(i)} - \delta_i, & \delta_i \geq c_2; \\ \omega^{(i)} - c_2, & \delta_i < c_2, \end{cases} \quad \omega_2^{(i)} = \begin{cases} \omega^{(i)} + \delta_i, & \delta_i \geq c_2; \\ \omega^{(i)} + c_2, & \delta_i < c_2, \end{cases}
\end{aligned}
$$

where c_1 and c_2 are small (here, we set $c_1 = 0.01$ and $c_2 = 0.001$). Using the current estimates of the parameters ω, the derivative of $\omega^{(i)}$ is

$$
\frac{\partial P(Y_T(\text{tx}) > k | A_T(0))}{\partial \omega^{(i)}} = \frac{f(\omega^{(-i)}, \omega_2^{(i)}) - f(\omega^{(-i)}, \omega_1^{(i)})}{2\delta_i},
$$

for $\text{tx} = 0, 1, 2$.

References

Press, W. H., Teukolsky, S. A., Vetterling, W. T., and Flannery, B. P. (1992). *Numerical Recipes in C: The Art of Scienific Computing.* Cambridge University Press, Cambridge, UK, second edition.