A Shrinkage Estimator for Spectral Densities: Web Appendix

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Result 1. *If* $Y \sim TN_0\left(\mu_Y, \sigma_Y^2\right)$ *with* $\mu_Y > 0$*, then* $E\left(Y^3\right) \leq 6\mu_Y\sigma_Y^2 + 2\mu_Y^3 + 8\sigma_Y^3 + 14\mu_Y^2\sigma_Y$.

Proof of Result 1 : Let $Y \sim TN_0\left(\mu_Y, \sigma_Y^2\right)$. $E\left(Y^3\right) = \left.\left(\frac{d^3 M_Y(t)}{dt^3}\right)\right|_{t=0}$ where

$$M_Y(t) = E\left(e^{ty}\right) = \int_0^\infty e^{ty}\frac{1}{\sigma_Y\sqrt{2\pi}}\left\{1-\Phi\left(-\frac{\mu_Y}{\sigma_Y}\right)\right\}^{-1}\exp\left\{-\frac{(y-\mu_Y)^2}{2\sigma_Y^2}\right\}dy =$$

$$\left\{1-\Phi\left(-\frac{\mu_Y}{\sigma_Y}\right)\right\}^{-1}\exp\left\{\left(2\mu_Y t\sigma_Y^2 + t^2\sigma_Y^2\right)/2\sigma_Y^2\right\}\left[1-\Phi\left\{-\frac{\left(\mu_Y + t\sigma_Y^2\right)}{\sigma_Y}\right\}\right].$$

Simple Calculus and basic algebra show that

$$\left.\frac{d^3}{dt^3}\left(\left\{1-\Phi\left(-\frac{\mu_Y}{\sigma_Y}\right)\right\}^{-1}\exp\left\{\left(2\mu_Y t\sigma_Y^2 + t^2\sigma_Y^2\right)/2\sigma_Y^2\right\}\left[1-\Phi\left\{-\frac{\left(\mu_Y + t\sigma_Y^2\right)}{\sigma_Y}\right\}\right]\right)\right|_{t=0} \leq$$

$$2\left(3\mu_Y\sigma_Y^2 + \mu_Y^3 + 4\sigma_Y^3 + 7\mu_Y^2\sigma_Y\right)$$

Result 2. *Let* Y *be a* $K \times 1$ *random vector such that* $Y \sim MVN\left(\mu, \Lambda\right)$ *and let* $\mu \sim TMVN\left(\beta, \Gamma\right)$*. Then* $Y|\Lambda, \beta, \Gamma \sim p_Y\left(y\right)$ *where*

$$p_Y\left(y\right) = \frac{c\left(\beta,\Gamma\right)}{\left(2\pi\right)^{\frac{K}{2}}\left|\Lambda+\Gamma\right|^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\left(y-\beta\right)'\left(\Lambda+\Gamma\right)^{-1}\left(y-\beta\right)\right\}\text{pr}\left(W \geq 0\right),$$

$c\left(\beta,\Gamma\right) = \left\{\text{pr}\left(\tilde{\mu} \geq 0\right)\right\}^{-1}$, $\tilde{\mu} \sim MVN\left(\beta, \Lambda\right)$*, and*

$$W \sim MVN\left\{\Lambda\left(\Lambda+\Gamma\right)^{-1}\beta + \Gamma\left(\Lambda+\Gamma\right)^{-1}Y, \left(\Lambda^{-1}+\Gamma^{-1}\right)^{-1}\right\}.$$

Proof of Result 2: First observe that since $\mu \sim TMVN(\beta, \Gamma)$, $\mu \sim p_\mu(\mu)$ where

$$p_\mu(\mu) = \begin{cases} \frac{c(\beta,\Gamma)}{(2\pi)^{\frac{K}{2}}|\Gamma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\mu-\beta)'\Gamma^{-1}(\mu-\beta)\right\} & \mu_i \geq 0 \ \forall \ i \\ 0 & o.w. \end{cases}$$

From this, it is clear that

$$\begin{aligned} p_Y(y) &= \int_{(\mathbb{R}^+)^K} \frac{1}{(2\pi)^{\frac{K}{2}}|\Lambda|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(y-\mu)'\Lambda^{-1}(y-\mu)\right\} \frac{c(\beta,\Gamma)}{(2\pi)^{\frac{K}{2}}|\Gamma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(\mu-\beta)'\Gamma^{-1}(\mu-\beta)\right\} \prod_{i=1}^{K} d\mu_i \\ &= \frac{c(\beta,\Gamma)}{(2\pi)^K|\Gamma|^{\frac{1}{2}}|\Lambda|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}\left(y'\Lambda^{-1}y + \beta\Gamma^{-1}\beta\right)\right\} \int_{(\mathbb{R}^+)^K} \exp\left[-\frac{1}{2}\left\{\mu'\left(\Lambda^{-1}+\Gamma^{-1}\right)\mu - 2\mu\left(\Lambda^{-1}y+\Gamma^{-1}\beta\right)\right\}\right] \prod_{i=1}^{K} d\mu_i. \end{aligned}$$

Letting $\gamma = \Lambda(\Lambda+\Gamma)^{-1}\beta + \Gamma(\Lambda+\Gamma)^{-1}y$, the above expression can be set equal to

$$\begin{aligned} &= \frac{c(\beta,\Gamma)}{(2\pi)^K|\Gamma|^{\frac{1}{2}}|\Lambda|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}\left(y'\Lambda^{-1}y+\beta'\Gamma^{-1}\beta\right) - \gamma'\left(\Lambda^{-1}+\Gamma^{-1}\right)\gamma\right\} \\ &\quad \times \int_{(\mathbb{R}^+)^K} \frac{(2\pi)^{\frac{K}{2}}\left|\left(\Lambda^{-1}+\Gamma^{-1}\right)\right|}{(2\pi)^{\frac{K}{2}}\left|\left(\Lambda^{-1}+\Gamma^{-1}\right)\right|} \exp\left\{-\frac{1}{2}(\mu-\gamma)'\left(\Lambda^{-1}+\Gamma^{-1}\right)(\mu-\gamma)\right\} \prod_{i=1}^{K} d\mu_i \\ &= \frac{c(\beta,\Gamma)}{(2\pi)^{\frac{K}{2}}|\Lambda+\Gamma|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2}(y-\beta)'(\Lambda+\Gamma)^{-1}(y-\beta)\right\} \mathrm{pr}(W \geq 0) \end{aligned}$$

Conclusion of Theorem 2: At the conclusion of Theorem 2, we claim that $\int_{\mathbb{R}^p} \frac{1}{c_1(\phi)^4} p(\phi) d\phi < \infty$ when an ARMA(p,0) model is specified. Following is the proof.

$$\begin{aligned} \int_{\mathbb{R}^p} \frac{1}{c_1(\phi)^4} p(\phi) d\phi &= \int_{\mathbb{R}^p} \left\{\sum_{j=1}^{F} P[j,1] g(\phi,\omega_j)\right\}^{-4} p(\phi) d\phi = \int_{\mathbb{R}^p} \left\{\sum_{j=1}^{F} \frac{P[j,1]}{|\phi(e^{-i\omega_j})|^{\frac{1}{2}}}\right\}^{-4} p(\phi) d\phi \\ &\leq \int_{\mathbb{R}^p} \left\{\sum_{j=1}^{F} \frac{P[j,1]}{(1+2\sum_{k=1}^{p}|\phi_k| + 4\sum_{l,k:l\neq k}|\phi_l\phi_k| + 2\sum_{k=1}^{p}\phi_k^2)^{\frac{1}{4}}}\right\}^{-4} p(\phi) d\phi \\ &= \int_{\mathbb{R}^p} \left(1+2\sum_{k=1}^{p}|\phi_k| + 4\sum_{l,k:l\neq k}|\phi_l\phi_k| + 2\sum_{k=1}^{p}\phi_k^2\right)\left(\sum_{j=1}^{F} P[j,1]\right)^{-4} p(\phi) d\phi. \end{aligned}$$

The last integral written above is finite given that $p(\phi)$ is proper and $\int_{\Re^p} |\phi_l\phi_k| p(\phi) d\phi < \infty$ for all pairs of autoregressive parameters. □

Result 6. $\hat{\tau}^2 \longrightarrow 0$ *in probability when the correct parametric model is specified.*

Proof of Result 6: From Result 2, it is clear that $p\left\{ \hat{f}_s(\omega)^{\frac{1}{4}} \middle| f_p(\omega)^{\frac{1}{4}}, \Sigma_I, \tau^2 \right\} =$

$$\frac{c\left\{f_p(\omega)^{\frac{1}{4}}, \tau^2\right\}}{(2\pi)^{\frac{K}{2}} \left|\Sigma_I + \tau^2 I\right|^{\frac{1}{2}}} \exp\left[-\frac{1}{2}\left\{\hat{f}_s(\omega)^{\frac{1}{4}} - f_p(\omega)^{\frac{1}{4}}\right\}' \left(\Sigma_I + \tau^2 I\right)^{-1} \left\{\hat{f}_s(\omega)^{\frac{1}{4}} - f_p(\omega)^{\frac{1}{4}}\right\}\right] \mathrm{pr}\,(U \geq 0),$$

where $U \sim N\left\{\Sigma_I\left(\Sigma_I + \tau^2 I\right)^{-1} f_p(\omega)^{\frac{1}{4}} + \tau^2\left(\Sigma_I + \tau^2 I\right)^{-1} \hat{f}_s(\omega)^{\frac{1}{4}}, \left(\Sigma_I^{-1} + \frac{1}{\tau^2} I\right)^{-1}\right\}$. Define $z(\omega) = \hat{f}_s(\omega)^{\frac{1}{4}} - f_p(\omega)^{\frac{1}{4}}$. We will begin by assuming that $z(\omega) \longrightarrow 0_{K\times 1}$ in probability; from Results 3 and 4, it is clear that this assumption is true. However, we assume that $\tau^2 \longrightarrow 0$ is not true. If $z(\omega) \longrightarrow 0_{K\times 1}$ in probability, it is impled that $z(\omega_l) \longrightarrow 0$ in probability for all $l \Longrightarrow \mathrm{pr}\,\{z(\omega_l) \geq \epsilon\} \longrightarrow 0$ for all $\epsilon > 0$ and for all l. Without loss of generality, let $l = 1$. Then

$$\begin{aligned}
&\mathrm{pr}\,\{z(\omega_1) \geq \epsilon\} \\
&= \int_{\epsilon}^{\infty} \int_{\mathbb{R}^{K-1}} \frac{c\left\{f_p(\omega)^{\frac{1}{4}}, \tau^2\right\}}{(2\pi)^{\frac{K}{2}} \left|\Sigma_I + \tau^2 I\right|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2} z' \left(\Sigma_I + \tau^2 I\right)^{-1} z\right\} \mathrm{pr}\,(U \geq 0) \left\{\prod_{j=2}^{K} dz(\omega_j)\right\} dz(\omega_1) \\
&\geq (.5)^K \int_{\epsilon}^{\infty} \int_{\mathbb{R}^{K-1}} \frac{1}{(2\pi)^{\frac{K}{2}} \left|\Sigma_I + \tau^2 I\right|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2} z' \left(\Sigma_I + \tau^2 I\right)^{-1} z\right\} \left\{\prod_{j=2}^{K} dz(\omega_j)\right\} dz(\omega_1) \\
&= (0.5)^K \left[1 - \Phi\left\{\frac{\epsilon}{\left(\Sigma[1,1] + \tau^2\right)^{\frac{1}{2}}}\right\}\right] > 0 \text{ for infinitely many } n
\end{aligned}$$

which contradicts $\mathrm{pr}\,\{z(\omega_l) \geq \epsilon\} \longrightarrow 0$ for all l. Note that the first inequality results since $c\left\{f_p(\omega)^{\frac{1}{4}}, \tau^2\right\} \geq 1$ and $\mathrm{pr}\,(U \geq 0) \geq 0.5^K$.

Result 7. $\hat{\tau}^2$ *is bounded away from 0 when the model is incorrectly specified.*

Proof of Result 7: This will be another proof by contradiction. Using the notation given in Result 6, assume that $\hat{f}_s(\omega)^{\frac{1}{4}} - f_p(\omega)^{\frac{1}{4}} = z(\omega) \longrightarrow 0$ in probability is not true, yet assume that $\tau^2 \longrightarrow 0$. If $z(\omega) \longrightarrow 0$ in probability is not true, then there exists some number $j \in \{1, 2, \ldots, K\}$ such that $\mathrm{pr}\,\{z(\omega_j) \geq \epsilon\} \geq \delta > 0$ for some ϵ and δ and for infinitely many n. Without loss of generality, assume that $j = 1$. Then

$$
\begin{aligned}
&\mathrm{pr}\left\{z\left(\omega_1\right) \geq \epsilon\right\} \\
&= \int_\epsilon^\infty \int_{\mathbb{R}^{K-1}} \frac{c\left\{f_p\left(\omega\right)^{\frac{1}{4}}, \tau^2\right\}}{\left(2\pi\right)^K \left|\Sigma_I + \tau^2 I\right|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2} z' \left(\Sigma_I + \tau^2 I\right)^{-1} z\right\} \mathrm{pr}\left(U \geq 0\right) \left\{\prod_{j=2}^K dz\left(\omega_j\right)\right\} dz\left(\omega_1\right) \\
&< (0.5)^{-K} \left[\int_\epsilon^\infty \int_{\mathbb{R}^{K-1}} \frac{1}{\left(2\pi\right)^K \left|\Sigma_I + \tau^2 I\right|^{\frac{1}{2}}} \exp\left\{-\frac{1}{2} z' \left(\Sigma_I + \tau^2 I\right)^{-1} z\right\} \left\{\prod_{j=2}^K dz\left(\omega_j\right)\right\} dz\left(\omega_1\right)\right]
\end{aligned}
$$

since $c\left\{f_p\left(\omega\right)^{\frac{1}{4}}, \tau^2\right\} \leq 0.5^{-K}$ and $\mathrm{pr}\left(U \geq 0\right) \leq 1$. Since it is known that $\Sigma_I \longrightarrow [0]_{K\times K}$ and it is assumed that $\tau^2 \longrightarrow 0$, it is clear that the quantity in curly brackets goes to 0. This contradicts the assumption that $z\left(\omega\right) \longrightarrow 0$ in probability is not true, however. As a result, we can conclude that $\tau^2 \longrightarrow 0$ is not true.